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Topological structures of adiabatic phase for multi-level quantum systems

Zheng-Xin Liu[1], **Xiao-Ting Zhou**[1], **Xin Liu**[2], **Xiong-Jun Liu**[3] **and Jing-Ling Chen**[1]

[1] Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China
[2] Department of Physics, Texas A&M University, College Station, TX 77843-4242, USA
[3] Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542

E-mail: liuzx@mail.nankai.edu.cn

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Abstract

The topological properties of adiabatic gauge fields for multi-level (three-level in particular) quantum systems are studied in detail. Similar to the result that the adiabatic gauge field for $SU(2)$ systems (e.g. two-level quantum system or angular momentum systems, etc) has a monopole structure, the curvature 2-forms of the adiabatic holonomies for $SU(3)$ three-level and $SU(3)$ eight-level quantum systems are shown to have monopole-like (for all levels) or instanton-like (for the degenerate levels) structures.

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1. Introduction

In the past 20 years, geometric phase has attracted much attention in quantum theory since Berry firstly showed that the adiabatic phase in a two-level system (or spin-half systems) has a monopole field strength in the parameter space [1]. Soon after that, Simon pointed out that Berry phase is the holonomy on the $U(1)$ fibre bundle formed by the Hamiltonian and its eigenstates, and the connection on the bundle is given by the parallel transport condition [2]. Furthermore, the geometric phase has been extended to processes which undergo non-adiabatic (AA phase) [3] or non-cyclic (Pancharatnam phase) [4] evolution. On the other hand, the non-Abelian adiabatic phase was first discussed by Zee and Wilczek [5]. The topological properties of non-Abelian Berry phase for $SO(2n+1)$ spinor were studied by Benedict *et al* [6]. In particular, Murakami *et al* discussed the $SU(2)$ holonomy of $SO(5)$ spinor and concluded that it is described by a Yang monopole at the degeneracy point [7].

We note that multi-level (level number $N \geqslant 3$) quantum systems appear in many application techniques, such as many-channel optical interferometry [8], quantum computation

[9] and quantum memory with electromagnetically induced transparency (EIT) [10, 11], etc. The study of the holonomy of such systems will have many interesting potential applications, e.g. *geometric* quantum computation & *geometric* quantum memory, and so on. Specifically, people can take advantage of these geometric features to the aim of quantum information processing, as the robustness of the Berry phase can result in a resilience against some kinds of decoherence sources [12]. As a result, the Berry phase in three-level systems ($SU(3)$ systems) has been discussed by many authors [13, 14]. The detection of geometric phase shift in a three-level system was achieved by Barry C Sanders *et al* through a three-channel optical interferometer [15]. Going beyond $SU(3)$, the Berry phase for general compact Lie groups was studied by E Strahov [16].

Although the adiabatic phase for $SU(3)$ systems has been widely studied, to our knowledge, the topology of the adiabatic gauge fields for multi-level systems has not been clearly investigated. In the present paper, we study in detail the topological properties of the adiabatic gauge fields for $SU(3)$ quantum systems. The development herein is outlined as follows. In section 2, we briefly review the monopole structure of $U(1)$ holonomy in $SU(2)$ system. In sections 3 and 4, we study the adiabatic gauge field for double degenerate three-level systems. We show that the curvature of the field for the non-degenerate level also has a monopole structure, while the non-Abelian curvature of the field for the degenerate level has a instanton-like structure. The topologies of the adiabatic gauge fields of the $SU(3)$ eight-level system and the non-degenerate three-level system are discussed in section 5. In the final section we conclude our results and present some further remarks. The detailed derivatives of some results are given in the appendix.

2. Monopole structures of the holonomy for $SU(2)$ systems

Before dealing with $SU(3)$ systems, let us review the geometric holonomy for $SU(2)$ systems. Firstly, we go over the non-degenerate case. The adiabatic phase for a spin $1/2$ particle in a magnetic field was firstly studied by Berry [1], where a monopole structure of the adiabatic gauge field strength was discovered. Here, for a general consideration, we envisage a particle with spin j interacting with an external magnetic field. There are three parameters in the Hamiltonian, namely, the magnetic field strength B ($B \geqslant 0$, can be seen as the radius of the parameter space) and two direction angles θ and φ. They span a three-dimensional Euclidian space $\mathbb{R}^3$. For the purpose of simplicity, we assume that B is independent of time. The time-dependent Hamiltonian $H(\mathbf{B})$ is given by the $SU(2)$ transform of $H_0 = \mu B \hat{J}_z$, which is the Hamiltonian in the rest frame (see the formula below). Because H_0 is invariant under the action of a $U(1)$ subgroup of $SU(2)$, which is generated by H_0 itself, we only need to consider the group element in the coset space $SU(2)/U(1) \cong \mathbb{C}P^1 \cong S^2$, which is known as the Bloch sphere. Thus the (co)adjoint orbit of H_0 with given non-zero B is a sphere S^2. So the parameter space can be seen as the collection of all the orbits. When $B = 0$, the orbit becomes a point, which is the origin of the parameter space $\mathbb{R}^3$.

$$
\begin{aligned}
&H_0(B) = \mu B \hat{J}_z, \qquad H(\mathbf{B}) = \mu \mathbf{B} \cdot \hat{\mathbf{J}} = U(\mathbf{B}) H_0 U(\mathbf{B})^\dagger, \\
&U(\mathbf{B}) = \exp\{-\mathrm{i}\theta[-\sin\varphi \hat{J}_x + \cos\varphi \hat{J}_y]\} = \exp\left\{-\frac{\theta}{2}[\mathrm{e}^{(-\mathrm{i}\varphi)} \hat{J}_+ - \mathrm{e}^{(\mathrm{i}\varphi)} \hat{J}_-]\right\} \qquad (1)\\
&\qquad = \exp\{-\eta \hat{J}_+\}(1 + |\eta|^2)^{\hat{J}_z} \exp\{\eta^* \hat{J}_-\},
\end{aligned}
$$

where $\mathbf{B}(t) = B(\sin\theta(t)\cos\varphi(t), \sin\theta \sin(t)\varphi(t), \cos\theta(t))$, and $\eta = \tan\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}$. The eigenstate of H_0 with eigenvalue $m\mu B$ is $\psi_0 = |m\rangle$, $m = -j, -j+1, \ldots j$, and the instantaneous eigenstate of $H(\mathbf{B})$ with the same eigenvalue is $\psi(\mathbf{B}) = U(\mathbf{B})\psi_0$. Then

$\{(\mathbf{B}, \psi(\mathbf{B})|H(\mathbf{B})\psi(\mathbf{B}) = E(\mathbf{B})\psi\mathbf{B})\}$ defines a line bundle over the parameter space. The Berry connection 1-form for this level is

$$A = \langle\psi(\mathbf{B})|\,\mathrm{d}|\psi(\mathbf{B})\rangle = \langle\psi_0|U^\dagger(\mathbf{B})\,\mathrm{d}U(\mathbf{B})|\psi_0\rangle \\ = A_\theta\,\mathrm{d}\theta + A_\varphi\,\mathrm{d}\varphi, \tag{2}$$

where $A_\theta = 0$, $A_\varphi = 2\mathrm{i}m\sin^2\frac{\theta}{2}$. And the curvature 2-form is

$$F = \mathrm{d}A = \langle\psi_0|\,\mathrm{d}U^\dagger(\mathbf{B}) \wedge \mathrm{d}U(\mathbf{B})|\psi_0\rangle = \frac{1}{2}\sum_{i,j=1}^{2} F_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{3}$$

where $x^1 = \theta$, $x^2 = \varphi$. A straightforward calculation gives $F_{\theta\varphi} = \frac{\mathrm{i}m}{2}\sin\theta$. The curvature (or the gauge field strength) can be written as a vector: $\mathbf{F} =^* (F_{ij}) = \frac{\mathrm{i}m\hat{\mathbf{B}}}{2B^2}$. The adiabatic phase for a cyclic evolution is equal to

$$\gamma_g = \mathrm{i}\oint_C \mathbf{A}\cdot\mathrm{d}\mathbf{l} = \mathrm{i}\int_{S,\partial S=C} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \mathrm{i}\int_{S,\partial S=C} \mathbf{F}\cdot\hat{\mathbf{B}}B^2\,\mathrm{d}\Omega = -m\Omega_{(S)}, \tag{4}$$

where C is the path of the magnetic field in the parameter space, S is the surface with edge C and $\Omega_{(S)}$ is the solid angle of the surface S. If we integral $\mathbf{F}$ through a surface surrounding the origin, we get the first Chern number:

$$\frac{\mathrm{i}}{2\pi}\oint_{S^2} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \frac{\mathrm{i}}{2\pi}\oint_{S^2} \mathbf{F}\cdot\hat{\mathbf{B}}B^2\,\mathrm{d}\Omega = -2m. \tag{5}$$

So, we can conclude that $\mathbf{F}$ describes a gauge field with a Dirac monopole at the origin, with its strength (i.e. the first Chern number) given by $-2m$.

Now, we turn to the degenerate case. If the Hamiltonian is not a linear but a quadratic combine of the $SU(2)$ generators like this form:

$$H_0 = (\mu B\hat{J}_z)^2, \qquad H(\mathbf{B}) = (\mu\mathbf{B}\cdot\hat{\mathbf{J}})^2 = U(\mathbf{B})H_0U(\mathbf{B})^\dagger. \tag{6}$$

Then for H_0 the spin state $\psi_1 = |m\rangle$ and $\psi_2 = |-m\rangle$ $(m \neq 0)$ have the same energy eigenvalue $m^2\mu^2B^2$, that is to say, this level is doubly degenerate. Thus in the adiabatic approximation, the connection 1-form is $A^{ik} = \langle\psi_i|U^\dagger(\mathbf{B})\,\mathrm{d}U(\mathbf{B})|\psi_k\rangle$, $(i, k = m, -m)$. From equation (1), it is easy to get that $A^{ik} = 0$ when $|m - (-m)| = 2m > 1$. So, if $m \neq 1/2$, the connection 1-form for the doubly degenerate Hilbert space is Abelian: $A = \frac{\mathrm{i}m\sigma_z}{B}\tan\frac{\theta}{2}B\sin\theta\,\mathrm{d}\varphi$, and it is equal to the direct sum of the connection 1-forms of the two components, which we have discussed above. If $m = 1/2$, the connection 1-form is non-Abelian:

$$A = -\frac{\mathrm{i}}{2}\left[-\cos\varphi\sigma_y + \left(j + \frac{1}{2}\right)\sin\varphi\sigma_x\right]\mathrm{d}\theta \\ + \frac{\mathrm{i}}{2}\left[\sin^2\frac{\theta}{2}\sigma_z + \left(j + \frac{1}{2}\right)\sin\theta(\cos\varphi\sigma_x + \sin\varphi\sigma_y)\right]\mathrm{d}\varphi \tag{7}$$

and the curvature 2-form is

$$F = \mathrm{d}A + A\wedge A = \left\{-\frac{\mathrm{i}}{2}\left[\left(j + \frac{1}{2}\right)^2 - 1\right]\frac{1}{B^2}\sigma_z\right\}B^2\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi, \tag{8}$$

which can also be written as a vector $\mathbf{F} = -\frac{\mathrm{i}}{2}\left[\left(j + \frac{1}{2}\right)^2 - 1\right]\frac{\mathbf{B}}{B^3}\sigma_z$. It is interesting that though the gauge potential is not diagonal, the field strength is diagonal. Similar to the non-degenerate case, the first Chern number is given by: $\frac{\mathrm{i}}{2\pi}\oint \mathrm{Tr}(\mathbf{F})\cdot\mathrm{d}\mathbf{S} = \mathrm{Tr}\left(-\mathrm{i}\left[\left(j + \frac{1}{2}\right)^2 - 1\right]\sigma_z\right) = 0$. Here each diagonal component of $\mathbf{F}$ is a Dirac monopole, yet the total 'flux'(described by the first Chern number) is zero. Specially, when $j = \frac{1}{2}$, the field strength $\mathbf{F}$ is zero.

As a natural extension, the adiabatic gauge field in $SU(m)(m \geqslant 3)$ systems has been developed by many authors [13, 14]. However, the topological structures of the adiabatic gauge fields are still not clear and deserve further investigation. In the following two sections, we focus on the topological properties of the adiabatic gauge fields for double-degenerate three-level quantum system (a special kind of $SU(3)$ system).

3. Coadjoint orbit of the doubly degenerate Hamiltonian for $SU(3)$ system and its geometry

The non-degenerate holonomy for the three-level system has been discussed in [13]. The authors identified the parameter space for a pure state (or its projection space) with the manifold of coset space $SU(3)/SU(2)$(or coset space $SU(3)/U(2)$), and find that the geometric phase is proportional to the generalized solid angle of the five-dimensional manifold. The adiabatic connection and Berry phase for doubly degenerate three-level systems have also been previously calculated [14]. With these results in mind, here we will further investigate the topological features of the Abelian and non-Abelian adiabatic gauge fields for $SU(3)$ systems. In this section, we first explore the structure of the coadjoint orbit (then the parameter space) of the Hamiltonian of the doubly degenerate three-level system, with which we can easily study the adiabatic gauge field for this system.

We choose the generators of $SU(3)$ as the eight Gell–Mann matrices, namely, λ_i, $i = 1, 2, \ldots, 8$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They obey the following rule:

$$\mathrm{Tr}\{\lambda_i \lambda_j\} = 2\delta_{ij}, \tag{9}$$

$$[\lambda_i, \lambda_j] = 2\mathrm{i} f_{ijk} \lambda_k, \tag{10}$$

where f_{ijk} are the structure constants:

$$f_{123} = 1, \qquad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \qquad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}. \tag{11}$$

The Hamiltonian and its eigenstates in the rest frame are given by (We assume that the first and the second energy levels are degenerate):

$$H_0 = \mathrm{diag}\{E_1, E_1, E_3\} = \frac{2E_1 + E_3}{3}\mathbf{I} + (E_1 - E_3)\frac{\lambda_8}{\sqrt{3}}, \tag{12}$$

$$|1\rangle = (1,0,0)^T, \qquad |2\rangle = (0,1,0)^T, \qquad |3\rangle = (0,0,1)^T, \tag{13}$$

where I is the unit matrix; we can omit this term in the Hamiltonian because it does not contribute to the geometric phase. So there is only one parameter left in H_0, namely, $R = \frac{(E_1-E_3)}{\sqrt{3}}$. In the adiabatic approximation, R is always positive or always negative (because when $R = 0$ the three eigenvalues are degenerate and the adiabatic approximation will not be satisfied). The sign of R does not influence the eigenstates. Generally, we take it to be positive and denote it as the radius in the parameter space, which is similar with the parameter B in $SU(2)$ case. Thus, the Hamiltonian in the rest frame can be rewritten as $H_0 = R\lambda_8$, and the time-dependent Hamiltonian is given by

$$\begin{aligned}
&H(\mathbf{R}) = U(\mathbf{R})H_0U(\mathbf{R})^\dagger,\\
&U(\mathbf{R}) = \mathrm{e}^{(\mathrm{i}\alpha\frac{\lambda_3}{2})}\,\mathrm{e}^{(\mathrm{i}\beta\frac{\lambda_2}{2})}\,\mathrm{e}^{(\mathrm{i}\gamma\frac{\lambda_3}{2})}\,\mathrm{e}^{(\mathrm{i}\theta\frac{\lambda_5}{2})}\,\mathrm{e}^{(\mathrm{i}a\frac{\lambda_3}{2})}\,\mathrm{e}^{(\mathrm{i}b\frac{\lambda_2}{2})}\,\mathrm{e}^{(\mathrm{i}c\frac{\lambda_3}{2})}\,\mathrm{e}^{(\mathrm{i}\phi\frac{\lambda_8}{2})},\\
&0 \leqslant \alpha, a \leqslant 2\pi;\quad 0 \leqslant \theta, \beta, b \leqslant \pi;\quad 0 \leqslant \gamma, c \leqslant 4\pi;\quad 0 \leqslant \phi \leqslant 2\sqrt{3}\pi.
\end{aligned} \tag{14}$$

Here we use the Euler's angles of $SU(3)$ introduced in [14, 17]. $\mathbf{R}$ represents the group parameters of $SU(3)$. It is easy to see that H_0 is invariant under the action of the subgroup $U(2)$ generated by $\lambda_1, \lambda_2, \lambda_3$ and λ_8, i.e. this $U(2)$ group is the isotropy group of the Hamiltonian. Therefore the coadjoint orbit of H_0 is the coset space $SU(3)/U(2) \cong \mathbb{C}P^2$ (see, for example, [18]). In the coset space, each group element $\bar{U}(\mathbf{R}) = \exp\{\mathrm{i}\alpha\frac{\lambda_3}{2}\}\exp\{\mathrm{i}\beta\frac{\lambda_2}{2}\}\exp\{\mathrm{i}\gamma\frac{\lambda_3}{2}\}\exp\{\mathrm{i}\theta\frac{\lambda_5}{2}\}$ corresponds to a new Hamiltonian $H(\mathbf{R})$. So, for a given radii R, the Hamiltonian's parameter space is a $\mathbb{C}P^2$ manifold. Together with the radii R, we get the five-dimensional parameter space of the Hamiltonian. Since $\mathbb{C}P^2 \subset S^7 \subset \mathbb{R}^8$ [19], here we give the eight-dimensional coordinates of the parameter space:

$$H(\mathbf{R}) = \sum_{i=1}^{8}\xi^i\lambda_i, \qquad \xi^i = \frac{1}{2}\mathrm{Tr}[H(\mathbf{R})\lambda_i], \tag{15}$$

$$\begin{aligned}
&\xi^1 = \frac{\sqrt{3}}{2}R\sin\beta\cos\alpha\sin^2\frac{\theta}{2}, &&\xi^2 = -\frac{\sqrt{3}}{2}R\sin\beta\sin\alpha\sin^2\frac{\theta}{2},\\
&\xi^3 = -\frac{\sqrt{3}}{2}R\cos\beta\sin^2\frac{\theta}{2}, &&\xi^4 = \frac{\sqrt{3}}{2}R\sin\theta\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2},\\
&\xi^5 = \frac{\sqrt{3}}{2}R\sin\theta\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}, &&\xi^6 = \frac{\sqrt{3}}{2}R\sin\theta\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2},\\
&\xi^7 = \frac{\sqrt{3}}{2}R\sin\theta\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}, &&\xi^8 = \frac{1}{4}R(3\cos\theta+1).
\end{aligned} \tag{16}$$

The north pole and the 'south sphere' of the manifold CP^2 are intrinsically important for our later discussion. For simplicity, we consider the unit $\mathbb{C}P^2$ with $R = 1$. Considering $-\frac{1}{2} \leqslant \xi^8 \leqslant 1$, the north pole is given by $H(\mathbf{R}) = \lambda_8 = \xi^i\lambda_i$, with $\xi^i = \delta_{i,8}$, and the south sphere is given by $\xi^8 = -\frac{1}{2}, \xi^4 = \xi^5 = \xi^6 = \xi^7 = 0, (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = \frac{3}{4}$. It is a sphere S_s^2 that describe the property of 'infinity' [20].

Now we give the metric on the $\mathbb{C}P^2$ manifold in the four-dimensional coordinates. Since in the Euclidian space $\mathbb{R}^8$, the metric is a unit matrix, so

$$\mathrm{d}s^2 = \sum_{i=1}^{8}\mathrm{d}\xi^i\,\mathrm{d}\xi^i = \sum_{m,n=1}^{4}g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n, \tag{17}$$

where $x^1 = \beta, x^2 = \alpha, x^3 = \gamma, x^4 = \theta$ (this gives a orientation on $\mathbb{C}P^2$); submitting (16) into (17) gives

$$(g_{mn}) = \begin{pmatrix} \frac{3}{4}\sin^2\frac{\theta}{2} & 0 & 0 & 0 \\ 0 & g_{22} & \frac{3}{16}\cos\beta\sin^2\theta & 0 \\ 0 & \frac{3}{16}\cos\beta\sin^2\theta & \frac{3}{16}\sin^2\theta & 0 \\ 0 & 0 & 0 & \frac{3}{4} \end{pmatrix},$$

where $g_{22} = \frac{3}{16}\sin^2\theta\cos^2\beta + \frac{3}{8}\sin^2\beta(1-\cos\theta)$. The volume is given by

$$V = \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\beta \int_0^{4\pi} \mathrm{d}\gamma \sqrt{\det(g)} = \frac{9\pi^2}{2}. \tag{18}$$

It is known that $\mathbb{C}P^2$ is a symplectic (Kähler) space. The symplectic (Kähler) form is given by [20]

$$\eta = \frac{1}{\sqrt{3}} f_{ijk}\xi_i \,\mathrm{d}\xi_j \,\mathrm{d}\xi_k, \tag{19}$$

which is invariant under $SU(3)$. Here η is normalized by $\langle \eta, \eta\rangle = 2$, where $\langle , \rangle$ is the obvious inner product for forms. The volume form is given by $\mathrm{d}V = \frac{1}{2}\eta^2$. In particular, η is self-dual. Using the formulae (11) and (16), one can write this symplectic form in the four-dimensional coordinate:

$$\eta = \frac{1}{2}\sum_{m,n=1}^{4} \eta_{mn}\,\mathrm{d}x^m \wedge \mathrm{d}x^n, \tag{20}$$

with

$$\eta_{12} = \frac{3}{4}\sin\beta\sin^2\frac{\theta}{2}, \qquad \eta_{24} = \frac{3}{8}\cos\beta\sin\theta, \qquad \eta_{34} = \frac{3}{8}\sin\theta, \quad \text{else} = 0. \tag{21}$$

It is easy to verify that the above form η is self-dual:

$$*\eta = \frac{1}{4\sqrt{\det(g^{-1})}} \sum_{a,b,c,d,m,n=1}^{4} g_{ac}g_{bd}\varepsilon^{abmn}\eta_{mn}\,\mathrm{d}x^c \wedge \mathrm{d}x^d = \eta. \tag{22}$$

Furthermore, the 2-forms like

$$\Omega_2 = f_{ijk}\,\mathrm{d}\xi_i\,\mathrm{d}\xi_j A_k(\xi) \qquad \text{with} \qquad \langle \Omega_2, \eta\rangle = 0 \tag{23}$$

span the space of anti-selfdual 2-forms. This form is also invariant under $SU(3)$.

4. Topology structures of the adiabatic $U(2)$ and $U(1)$ fields for the three-level system

In the above section, we have introduced the parameter space of the Hamiltonian in the double-degenerate case. Now we shall probe into the features of the adiabatic gauge fields on the parameter space. To study the evolution of the Hamiltonian and its eigenfunctions, we only need to consider the following 4-parameter group elements:

$$\begin{aligned} &\bar{U}(\mathbf{R}) = \exp\left\{\mathrm{i}\alpha\frac{\lambda_3}{2}\right\}\exp\left\{\mathrm{i}\beta\frac{\lambda_2}{2}\right\}\exp\left\{\mathrm{i}\gamma\frac{\lambda_3}{2}\right\}\exp\left\{\mathrm{i}\theta\frac{\lambda_5}{2}\right\}, \\ &|j(\mathbf{R})\rangle = \bar{U}(\mathbf{R})|j\rangle, \\ &0 \leqslant \alpha \leqslant 2\pi; \quad 0 \leqslant \beta, \theta \leqslant \pi; \quad 0 \leqslant \gamma \leqslant 4\pi; \quad j = 1, 2, 3. \end{aligned} \tag{24}$$

where $|1\rangle$ and $|2\rangle$ are degenerate and span the eigenspace of level E_1, and $|3\rangle$ is the eigenstate of level E_3, and $|j(\mathbf{R})\rangle$ is the jth instantaneous eigenstate of the $H(\mathbf{R}) = \bar{U}(\mathbf{R})H_0\bar{U}(\mathbf{R})^\dagger$. Now we have two adiabatic gauge fields corresponding to the two levels. In other words, they form two complex bundles over the manifold $\mathbb{C}P^2$, and the structure groups are $U(1)$ and $U(2)$ respectively. These two bundles are associated with two principal bundles: one is the $U(1)$ principal bundle known as the Hopf bundle $P(\mathbb{C}P^2, U(1))$, and the other is the $U(2)$ principal bundle which equals to the group manifold $SU(3) = P(\mathbb{C}P^2, U(2))$,

$$\begin{array}{ccc} U(2) & \longrightarrow & SU(3) \\ & & \downarrow \\ & & \mathbb{C}P^2 \end{array}$$

The connection 1-forms of the two gauge fields are given below:

$$\begin{aligned} A^{(E_1)ij} &= \langle i(\mathbf{R})|d|j(\mathbf{R})\rangle \\ &= \langle i|\bar{U}^\dagger(\mathbf{R})\mathrm{d}\bar{U}(\mathbf{R})|j\rangle, \qquad i, j = 1, 2, \end{aligned} \tag{25}$$

$$A^{(E_3)} = \langle 3|\bar{U}^\dagger(\mathbf{R})\mathrm{d}\bar{U}(\mathbf{R})|3\rangle. \tag{26}$$

A straightforward calculation gives ($A = \sum_{m=1}^{4} A_m \mathrm{d}x^m$, where $x^1 = \beta, x^2 = \alpha, x^3 = \gamma, x^4 = \theta$):

$$\begin{aligned} A_1^{(E_1)} &= \frac{\mathrm{i}}{4}\left[-\sin\left(\frac{\theta}{2}+\gamma\right) + \sin\left(\frac{\theta}{2}-\gamma\right)\right]\sigma_x + \frac{\mathrm{i}}{4}\left[\cos\left(\frac{\theta}{2}-\gamma\right) - \cos\left(\frac{\theta}{2}+\gamma\right)\right]\sigma_y, \\ A_2^{(E_1)} &= \frac{\mathrm{i}}{4}\left[-\sin^2\frac{\theta}{2}\cos\beta\mathbf{I} + \left(\cos^2\frac{\theta}{2}+1\right)\cos\beta\sigma_z\right] + \frac{\mathrm{i}}{8}\left[\sin\left(\beta-\frac{\theta}{2}+\gamma\right)\right. \\ &\quad \left. + \sin\left(\beta-\frac{\theta}{2}-\gamma\right) + \sin\left(\beta+\frac{\theta}{2}+\gamma\right) + \sin\left(\beta+\frac{\theta}{2}-\gamma\right)\right]\sigma_x \\ &\quad + \frac{\mathrm{i}}{8}\left[-\cos\left(\beta-\frac{\theta}{2}+\gamma\right) + \cos\left(\beta-\frac{\theta}{2}-\gamma\right) - \cos\left(\beta+\frac{\theta}{2}+\gamma\right)\right. \\ &\quad \left. + \cos\left(\beta+\frac{\theta}{2}-\gamma\right)\right]\sigma_y, \\ A_3^{(E_1)} &= \frac{\mathrm{i}}{4}\left[-\sin^2\frac{\theta}{2}\mathbf{I} + \left(1+\cos^2\frac{\theta}{2}\right)\sigma_z\right], \qquad A_4^{(E_1)} = 0; \\ A_1^{(E_3)} &= 0, \qquad A_2^{(E_3)} = \frac{\mathrm{i}}{2}\cos\beta\sin^2\frac{\theta}{2}, \qquad A_3^{(E_3)} = \frac{\mathrm{i}}{2}\sin^2\frac{\theta}{2}, \qquad A_4^{(E_3)} = 0. \end{aligned} \tag{27}$$

It is interesting that the connection form for level E_1 has a $U(1)$ component (the terms with unit matrix $\mathbf{I}$), which is proportional to the connection form of level E_3. The above connection forms are defined on $\mathbb{C}P^2$. We can extend the field to the five-dimensional parameter space with the connection along the radii $A_5 = A_R = 0, (x^5 = R)$. Then one can calculate the geometric phase through a integral of these 1-forms along a path in the parameter space: $\gamma_g = \mathrm{P}\exp\left\{\oint_c A\right\}$, here P represents path ordering and c is a path in the parameter space. Now we focus on the topological structures of the adiabatic gauge fields. The curvature 2-forms are given by

$$F = \mathrm{d}A + A\wedge A = \frac{1}{2}\sum_{m,n=1}^{4} F_{mn}\,\mathrm{d}x^m\wedge\mathrm{d}x^n. \tag{28}$$

The components of the curvatures are given below:

$$\begin{aligned}
F_{12}^{(E_1)} &= \frac{\mathrm{i}}{4}\left[\sin\beta\sin^2\frac{\theta}{2}(\mathbf{I}-\sigma_z)+\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}(\cos\gamma\sigma_x+\sin\gamma\sigma_y)\right],\\
F_{13}^{(E_1)} &= \frac{\mathrm{i}}{4}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}[\cos\gamma\sigma_x+\sin\gamma\sigma_y],\\
F_{14}^{(E_1)} &= \frac{\mathrm{i}}{4}\sin\frac{\theta}{2}[\cos\gamma\sigma_y-\sin\gamma\sigma_x],\\
F_{23}^{(E_1)} &= \frac{\mathrm{i}}{4}\sin\beta\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}[-\cos\gamma\sigma_y+\sin\gamma\sigma_x],\\
F_{24}^{(E_1)} &= \frac{\mathrm{i}}{4}\left[\frac{1}{2}\cos\beta\sin\theta(\mathbf{I}+\sigma_z)+\sin\beta\sin\frac{\theta}{2}(\cos\gamma\sigma_x+\sin\gamma\sigma_y)\right],\\
F_{34}^{(E_1)} &= \frac{\mathrm{i}}{8}\sin\theta[\mathbf{I}+\sigma_z];\\
F_{12}^{(E_3)} &= -\frac{\mathrm{i}}{2}\sin\beta\sin^2\frac{\theta}{2}, \qquad F_{24}^{(E_3)} = -\frac{\mathrm{i}}{4}\cos\beta\sin\theta,\\
F_{34}^{(E_3)} &= -\frac{\mathrm{i}}{4}\sin\theta, \qquad F_{23}^{(E_3)} = F_{13}^{(E_3)} = F_{14}^{(E_3)} = 0.
\end{aligned} \tag{29}$$

Now we can see that the 2-form $F^{(E1)}$ can be decomposed into an $SU(2)$ part and an $U(1)$ part $F^{(E_1)} = F^{E_1[U(1)]} + F^{E_1[SU(2)]}$. The $U(1)$ part $F^{E_1[U(1)]}$ together with the 2-form $F^{(E_3)}$ is proportional to the self-dual form given in equations (20) and (21): $F^{E_1[U(1)]} = \frac{\mathrm{i}}{3}\mathbf{I}\eta$, $F^{(E_3)} = -\frac{2\mathrm{i}}{3}\eta$, whereas the $SU(2)$ part $F^{E_1[SU(2)]}$ satisfies equation (23) and is anti-selfdual.

The de Rham cohomology of $\mathbb{C}P^2$ is given by $H^2(\mathbb{C}P^2) = \mathbb{R}\eta$ and $H^4(\mathbb{C}P^2) = \mathbb{R}\eta^2$. The integer cohomology $H^{2*}(\mathbb{C}P^2;\mathbb{Z})$ is generated by $\omega = \frac{\eta}{3\pi}$ [20], i.e.

$$\int_{S_s^2}\omega = \int_{CP^2}\omega\wedge\omega = 1, \tag{30}$$

where the 2- and 4-cycles are represented by the south sphere S_s^2 and $\mathbb{C}P^2$ itself. This non-trivial topological property of $\mathbb{C}P^2$ leads to the non-trivial properties of the gauge fields on it. For bundles over $\mathbb{C}P^2$, the first and the second Chern classes are well defined: $c_1 = \frac{\mathrm{i}}{2\pi}\,\mathrm{Tr}\,F$, and $c_2 = \frac{1}{8\pi^2}[\mathrm{Tr}(F\wedge F) - \mathrm{Tr}\,F\wedge\mathrm{Tr}\,F]$. And the Chern numbers are given by

$$c_1 = \frac{\mathrm{i}}{2\pi}\int_{S_s^2}\mathrm{Tr}\,F, \qquad c_2 = \frac{1}{8\pi^2}\int_{CP^2}[\mathrm{Tr}(F\wedge F) - \mathrm{Tr}\,F\wedge\mathrm{Tr}\,F]. \tag{31}$$

One can easily get the Chern numbers for the $U(1)$ bundle and the $U(2)$ bundle:

$$c_1(E_3) = 1, \qquad c_1(E_1) = -1, \qquad c_2(E_1) = 1. \tag{32}$$

This indicates that the $U(1)$ gauge field (for level $E(3)$) has a monopole with 'charge' 1, and the $U(2)$ gauge field (for level E_3) has a instanton-like structure. The latter is not a usual instanton, because the field strength is neither self-dual nor anti-selfdual. However, just like the usual instanton, the 'action' $I^{(E_3)}$ of the $U(2)$ gauge field is coincident with the second Chern number c_2 [20]:

$$\begin{aligned}
c_2 &= \frac{1}{8\pi^2}\int_{CP^2}[\mathrm{Tr}(F^{(E_3)}\wedge F^{(E_3)}) - \mathrm{Tr}\,F^{(E_3)}\wedge\mathrm{Tr}\,F^{(E_3)}]\\
&= -\frac{1}{8\pi^2}\int_{CP^2}\mathrm{Tr}[F^{(E_3)}\wedge_* F^{(E_3)}]\\
&\propto I^{(E_3)}.
\end{aligned} \tag{33}$$

It is known that $\mathbb{C}P^2$ is not a spin manifold but a spinc manifold. There is no global $SU(2)$ spinor section on $\mathbb{C}P^2$, but here we have induced a bundle with a wavefunction section whose structure group is $U(2)$. It is interesting that this bundle has the properties of both monopole (described by c_1) and instanton (described by c_2).

5. Adiabatic gauge fields for the degenerate $SU(3)$ eight-level system and the completely non-degenerate three-level system

Similar to the case that the $U(1)$ monopole of $SU(2)$ system can be extended to high-dimensional representation of group $SU(2)$, the $U(2)$ bundle can also be extended to high-dimensional representation of group $SU(3)$. As an example, we study here the adjoint representation of $SU(3)$, which describes an eight-level system. For this we chose the generators as $(\Lambda_i)_{jk} = f_{ijk}$, where f_{ijk} is given by equation (11). Since Λ_3 and Λ_8 are not diagonal, we can diagonalize them simultaneously by a unitary transformation: $(\Lambda_i)_{jk} = \mathrm{i} f_{ijk}$ (see the appendix).

Since the geometric phase for a general $SU(3)$ eight-level system has no new topological structure, we only consider the Hamiltonian with an $SU(2)$ symmetry (then the eigenvalues have an extra degeneracy). Similar to equation (12), the Hamiltonian in the rest frame is $H_0 = R\Lambda_8'$. And the time-dependent Hamiltonian is given by

$$\begin{aligned} &H(\mathbf{R}) = \bar{U}(\mathbf{R}) H_0 \bar{U}(\mathbf{R})^{\dagger}, \\ &\bar{U}(\mathbf{R}) = \exp\left\{\mathrm{i}\alpha \frac{\Lambda_3'}{2}\right\} \exp\left\{\mathrm{i}\beta \frac{\Lambda_2'}{2}\right\} \exp\left\{\mathrm{i}\gamma \frac{\Lambda_3'}{2}\right\} \exp\left\{\mathrm{i}\theta \frac{\Lambda_5'}{2}\right\}. \end{aligned} \tag{34}$$

The parameter space of the Hamiltonian for a given R is also $\mathbb{C}P^2$. The Hamiltonian has three eigenvalues: $\pm\frac{\sqrt{3}}{2}R$ and 0. The former two levels are both doubly degenerate, and the latter is four-fold degenerate. Now we have two $U(2)$ adiabatic fields and a quasi-$U(4)$ gauge field (because the field strength of the four-fold degenerate space is traceless and reducible). The gauge potential and the strength of the fields are given in the appendix . The first Chern number and second Chern number are given by ($\pm$ and 0 symbols the energy level)

$$\begin{aligned} c_1^{(-)} &= \frac{\mathrm{i}}{2\pi} \int_{S_s^2} \mathrm{Tr}\, F^{(-)} = 3, \\ c_1^{(0)} &= \frac{\mathrm{i}}{2\pi} \int_{S_s^2} \mathrm{Tr}\, F^{(0)} = 0, \\ c_1^{(+)} &= \frac{\mathrm{i}}{2\pi} \int_{S_s^2} \mathrm{Tr}\, F^{(+)} = -3; \\ c_2^{(-)} &= \frac{1}{8\pi^2} \int_{CP^2} [\mathrm{Tr}(F^{(-)} \wedge F^{(-)}) - \mathrm{Tr}\, F^{(-)} \wedge \mathrm{Tr}\, F^{(-)}] = 3, \\ c_2^{(0)} &= \frac{\mathrm{i}}{8\pi^2} \int_{CP^2} [\mathrm{Tr}(F^{(0)} \wedge F^{(0)}) - \mathrm{Tr}\, F^{(0)} \wedge \mathrm{Tr}\, F^{(0)}] = 3, \\ c_2^{(+)} &= \frac{\mathrm{i}}{8\pi^2} \int_{CP^2} [\mathrm{Tr}(F^{(+)} \wedge F^{(+)}) - \mathrm{Tr}\, F^{(+)} \wedge \mathrm{Tr}\, F^{(+)}] = 3. \end{aligned} \tag{35}$$

So, for the adiabatic $U(2)$ fields with level $\pm\frac{\sqrt{3}}{2}R$, we have got the similar result to our former discussion. However, for level 0, it is very interesting that the strength of the four-dimensional gauge field can be divided into an $U(1)$ field whose field strength is zero and a three-component $SU(2)$ gauge field or spin-1 gauge field (to get this result one only needs to make a global similarity transformation, since the third row and the third column of $F_{mn}^{(0)}$

are zero). This $SU(2)$ gauge filed is anti-self dual with finite 'Yang–Mills Action', which is described by the second Chern number. So we can say that we have got an instanton on $\mathbb{C}P^2$. Remembering that the parameter space is a five-dimensional manifold spanned by the radii R and a four-dimensional 'surface' $\mathbb{C}P^2$, analogous to the $SU(2)$ Yang monopole on S^4 or $\mathbb{R}^5$ (which is described by the second Chern number) [7, 21], we can also say that we have got a Yang-monopole-like gauge field on the parameter space. This result is based on the factor that $H^4(\mathbb{C}P^2;\mathbb{Z})$ is non-zero.

Above we have discussed the $SU(3)$ systems with an $U(2)$ symmetry (accordingly there are extra degeneracy). For non-degenerate three-level systems, the Hamiltonian has the symmetry of $U(1)\otimes U(1)$(generated by λ_3 and λ_8), and the parameter space is given by the coset $SU(3)/U(1)\otimes U(1) = F_2$, which is a flag space. This flag space is topologically a S^2 bundle over $\mathbb{C}P^2$ [22]. The group elements in the coset space and the Hamiltonian are given by

$$H_0 = R_3\lambda_3 + R_8\lambda_8, \qquad H(\mathbf{R}) = \widetilde{U}(\mathbf{R})H_0\widetilde{U}^{\dagger}(\mathbf{R}),$$
$$\widetilde{U}(\mathbf{R}) = \bar{U}(\mathbf{R})\exp\left\{\mathrm{i}\varphi_2\frac{\lambda_3}{2}\right\}\exp\left\{\mathrm{i}\varphi_1\frac{\lambda_2}{2}\right\}, \qquad (0\leqslant\varphi_1<\pi, 0\leqslant\varphi_2<2\pi), \tag{36}$$

where $\bar{U}(\mathbf{R})$ is given by equation (24). The adiabatic gauge fields are all $U(1)$ fields. There is some discussion about the property of these fields in [13]; here we reinspect it from another point of view. In fact, the three eigenstates are equivalent, that is to say, we can change the order of the three eigenvalues of H_0 via a similarity transformation. Therefore, we only need to consider the adiabatic field for the third eigenstate $(0, 0, 1)^T$. Since the projective space for a three-dimensional Hilbert space is just $\mathbb{C}P^2$, the parameter space for the Hamiltonian gives redundant information for its eigenstate (the angles φ_1 and φ_2 in $\widetilde{U}(\mathbf{R})$ are redundant), thus the components of the gauge potential along these variables are zero. The non-zero components of the adiabatic gauge potential are the same as $A^{(E_3)}$ (see equation (27)), and the curvature 2-form F is the same as $F^{(E_3)}$ (see equation (29)).

Since the second Betti number (the dimension of the second cohomology group) of the flag space F_2 is $b_2 = 2$ [23], and we can easily find two close 2-forms on F_2: one is η which is defined by equations (20) and (21), the other is $\tau = \sin\varphi_1\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2$, which is the volume form on the fibre S^2, the second de Rham cohomology of F_2 can be written as $H^2(F_2) = \mathbb{R}\eta + \mathbb{R}\tau$. Since the parameters on the fibre S^2 are redundant, the 2-form τ does not give a topological number for the bundles over F_2. And the self-dual form η corresponds to the first Chern number $\left(F = F^{(E_3)} = -\frac{2\mathrm{i}}{3}\eta\right)$:

$$c_1 = \frac{\mathrm{i}}{2\pi}\oint_{S_s^2} F = 1. \tag{37}$$

So the $U(1)$ monopole defined on $\mathbb{C}P^2$ in the degenerate case also exists on the parameter space in the non-degenerate case.

6. Conclusions and further discussions

In conclusion, we have discussed the topology structure of the adiabatic gauge field in $SU(3)$ quantum systems. For the twofold-degenerate three-level system (the Hamiltonian has an $U(2)$ symmetry), we find that on the parameter space ($\mathbb{C}P^2$ or the five-dimensional space), the curvature 2-form for the non-degenerate level has the feature of $U(1)$ monopole, while the curvature for the degenerate level has either an instanton-like or a monopole-like structure. It is interesting that for the $SU(3)$ adjoint representation system (eight-level system) whose

Hamiltonian has an $U(2)$ symmetry, the adiabatic gauge field for the four-fold-degenerate level has a Yang-monopole-like structure. The adiabatic gauge fields for a non-degenerate three-level system have also been discussed and are shown to have monopole structures.

It will be interesting to generalize the present results to a general multi-level system, say, $SU(N)(N > 3)$ systems. In the defining representation, when the Hamiltonian for $SU(N)$ system has a $U(N-1)$ symmetry, we get a $U(N-1)$ gauge field and a $U(1)$ field on the parameter space $\mathbb{C}P^{(N-1)} \cong SU(N)/U(N-1)$, one can expect that the former may have an instanton-like structure while the latter may have a monopole-like structure. General N-level quantum systems have been employed in many applications to quantum information processing and quantum information storage [24]. Therefore, to study the geometric phase and topological properties in these systems will be an interesting issue, and will be studied in our next investigation.

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Appendix

The generators of the adjoint representation of $SU(3)$ are given by $(\Lambda_i)_{jk} = f_{ijk}$. Since Λ_3 and Λ_8 are not diagonal, we can diagonalize them by a matrix V with $\Lambda_3' = V\lambda_3 V^\dagger$, $\Lambda_8' = V\lambda_8 V^\dagger$:

$$V = \begin{pmatrix} 0 & 0 & -1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3' = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

$$\Lambda_8' = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Other six generators $\Lambda_i' = V\Lambda_i V^\dagger$ are given below:

$$\Lambda_1' = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & -\frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\Lambda_2' = \begin{pmatrix} 0 & \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 \end{pmatrix},$$

$$\Lambda_4' = \begin{pmatrix} 0 & 0 & 0 & -\frac{\mathrm{i}}{\sqrt{8}} & -\mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{\mathrm{i}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{\sqrt{8}} & 0 \\ \mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\sqrt{\frac{3}{8}} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{8}} & \mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_5 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{2} \\ -\frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{8}} & 0 \\ -\sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{8}} & 0 \\ 0 & \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_6 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{8}} & -\mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{i}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{8}} \\ 0 & \mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\sqrt{\frac{3}{8}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{i}}{\sqrt{8}} & \mathrm{i}\sqrt{\frac{3}{8}} & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_7 = \begin{pmatrix} 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{8}} \\ 0 & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{8}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 \end{pmatrix}.$$

The following are the components of the gauge potential and the strength of the adiabatic gauge field for the $SU(3)$ adjoint representation system, where +, − and 0 label the levels of the Hamiltonian.

$$A_1^{(-)} = \begin{pmatrix} 0 & -\frac{1}{4}(\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}+\gamma)} + \mathrm{e}^{\mathrm{i}(-\frac{\theta}{2}+\gamma)}) \\ \frac{1}{4}(\mathrm{e}^{-\mathrm{i}(\frac{\theta}{2}+\gamma)} + \mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\gamma)}) & 0 \end{pmatrix},$$

$$A_2^{(-)} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\cos\beta\cos\theta & \frac{\mathrm{i}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ \frac{\mathrm{i}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & \frac{\mathrm{i}}{4}\cos\beta(3-\cos\theta) \end{pmatrix},$$

$$A_3^{(-)} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\cos\theta & 0 \\ 0 & \frac{\mathrm{i}}{4}(3-\cos\theta) \end{pmatrix}, \qquad A_4^{(-)} = 0,$$

$$A_1^{(0)} = \begin{pmatrix} 0 & -\frac{\sqrt{2}\mathrm{i}}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ -\frac{\sqrt{2}\mathrm{i}}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}\mathrm{i}}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}\mathrm{i}}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 \end{pmatrix},$$

$$A_2^{(0)} = \begin{pmatrix} -\frac{\mathrm{i}}{4}\cos\beta(3+\cos\theta) & -\frac{\sqrt{2}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ \frac{\sqrt{2}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2}\sin\beta\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & \frac{\mathrm{i}}{4}\cos\beta(3+\cos\theta) \end{pmatrix},$$

$$A_3^{(0)} = \begin{pmatrix} -\frac{\mathrm{i}}{4}(3+\cos\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{4}(3+\cos\theta) \end{pmatrix}, \qquad A_4^{(0)} = 0,$$

$$A_m^{(+)} = A_m^{*(-)}, \qquad m = 1, 2, 3, 4$$

$$F_{12}^{(-)} = \begin{pmatrix} -\mathrm{i}\sin\beta\sin^2\frac{\theta}{2} & \frac{\mathrm{i}}{8}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ \frac{\mathrm{i}}{8}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & -2\mathrm{i}\sin\beta\sin^2\frac{\theta}{2} \end{pmatrix},$$

$$F_{13}^{(-)} = \begin{pmatrix} 0 & \frac{\mathrm{i}}{8}\sin^2\frac{\theta}{2}(\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}+\gamma)} + \mathrm{e}^{\mathrm{i}(-\frac{\theta}{2}+\gamma)}) \\ \frac{\mathrm{i}}{8}\sin^2\frac{\theta}{2}(\mathrm{e}^{-\mathrm{i}(\frac{\theta}{2}+\gamma)} + \mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\gamma)}) & 0 \end{pmatrix},$$

$$F_{14}^{(-)} = \begin{pmatrix} 0 & -\frac{1}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ \frac{1}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 \end{pmatrix},$$

$$F_{23}^{(-)} = \begin{pmatrix} 0 & \frac{1}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\frac{1}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 \end{pmatrix},$$

$$F_{24}^{(-)} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\cos\beta\sin\theta & \frac{\mathrm{i}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ \frac{\mathrm{i}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & -\frac{\mathrm{i}}{4}\cos\beta\sin\theta \end{pmatrix}, \qquad F_{34}^{(-)} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\sin\theta & 0 \\ 0 & -\frac{\mathrm{i}}{4}\sin\theta \end{pmatrix},$$

$$F_{12}^{(0)} = \begin{pmatrix} \frac{\mathrm{i}}{2}\sin\beta\sin^2\frac{\theta}{2} & -\frac{\sqrt{2}}{4}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ \frac{\sqrt{2}}{4}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}}{4}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4}\cos\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & -\frac{\mathrm{i}}{2}\sin\beta\sin^2\frac{\theta}{2} \end{pmatrix},$$

$$F_{13}^{(0)} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{4}\cos\theta\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ \frac{\sqrt{2}}{4}\cos\theta\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}}{4}\cos\theta\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4}\cos\theta\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 \end{pmatrix},$$

$$F_{14}^{(0)} = \begin{pmatrix} 0 & -\frac{\sqrt{2}\mathrm{i}}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ -\frac{\sqrt{2}\mathrm{i}}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}\mathrm{i}}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}\mathrm{i}}{4}\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 \end{pmatrix},$$

$$F_{23}^{(0)} = \begin{pmatrix} 0 & \frac{\sqrt{2}\mathrm{i}}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ \frac{\sqrt{2}\mathrm{i}}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & \frac{\sqrt{2}\mathrm{i}}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}\mathrm{i}}{4}\sin\beta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 \end{pmatrix},$$

$$F_{24}^{(0)} = \begin{pmatrix} -\frac{\mathrm{i}}{4}\cos\beta\sin\theta & -\frac{\sqrt{2}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \\ \frac{\sqrt{2}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & 0 & -\frac{\sqrt{2}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4}\sin\beta\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\gamma} & 0 & \frac{\mathrm{i}}{4}\cos\beta\sin\theta \end{pmatrix},$$

$$F_{34}^{(0)} = \begin{pmatrix} -\frac{\mathrm{i}}{4}\sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{4}\sin\theta \end{pmatrix},$$

$$F_{mn}^{(+)} = F_{mn}^{*(-)} \qquad m, n = 1, 2, 3, 4.$$

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